10. Prove that two diagonalizable matrices are simultaneously diagonalizable, that is, that there is an invertible matrix P such that PAP^{-1} and PBP^{-1} are both diagonal, if and only if $AB = BA$.

***11.** Let A be a finite abelian group, and let $\varphi: A \longrightarrow \mathbb{C}^\times$ be a homomorphism which is not the trivial homomorphism ($\varphi(x) = 1$ for all x). Prove that $\sum_{a \in A} \varphi(a) = 0$.

12. Let A be an $m \times n$ matrix with coefficients in a ring R, and let $\varphi: R^n \longrightarrow R^m$ be left multiplication by A. Prove that the following are equivalent:
(i) φ is surjective;
(ii) the determinants of the $m \times m$ minors of A generate the unit ideal;
(iii) A has a right inverse, a matrix B with coefficients in R such that $AB = I$.

***13.** Let $(v_1, \ldots, v_m)$ be generators for an R-module V, and let J be an ideal of R. Define JV to be the set of all finite sums of products av, $a \in J$, $v \in V$.
(a) Show that if $JV = V$, there is an $n \times n$ matrix A with entries in J such that $(v_1, \ldots, v_m)(I - A) = 0$.
(b) With the notation of (a), show that $\det(I - A) = 1 + \alpha$, where $\alpha \in J$, and that $\det(I - A)$ annihilates V.
(c) An R-module V is called *faithful* if $rV = 0$ for $r \in R$ implies $r = 0$. Prove the *Nakayama Lemma:* Let V be a finitely generated, faithful R-module, and let J be an ideal of R. If $JV = V$, then $J = R$.
(d) Let V be a finitely generated R-module. Prove that if $MV = V$ for all maximal ideals M, then $V = 0$.

***14.** We can use a pair $x(t), y(t)$ of complex polynomials in t to define a complex path in $\mathbb{C}^2$, by sending $t \rightsquigarrow (x(t), y(t))$. They also define a homomorphism $\varphi: \mathbb{C}[x, y] \longrightarrow \mathbb{C}[t]$, by $f(x, y) \rightsquigarrow f(x(t), y(t))$. This exercise analyzes the relationship between the path and the homomorphism. Let's rule out the trivial case that $x(t), y(t)$ are both constant.
(a) Let S denote the image of φ. Prove that S is isomorphic to the quotient $\mathbb{C}[x, y]/(f)$, where $f(x, y)$ is an irreducible polynomial.
(b) Prove that t is the root of a monic polynomial with coefficients in S.
(c) Let V denote the variety of zeros of f in $\mathbb{C}^2$. Prove that for every point $(x_0, y_0) \in V$, there is a $t_0 \in \mathbb{C}$ such that $(x_0, y_0) = (x(t_0), y(t_0))$.